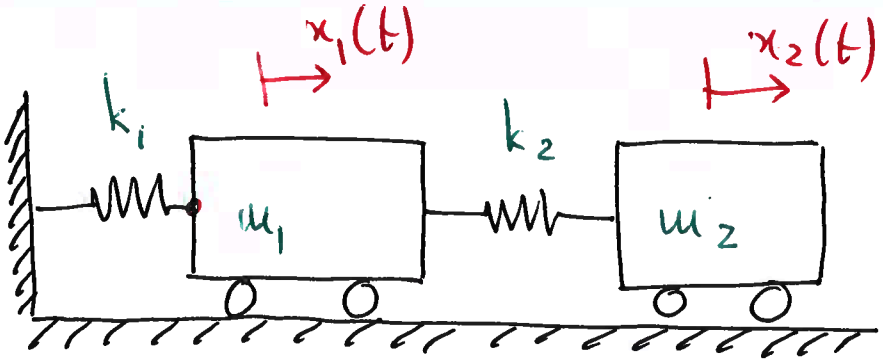
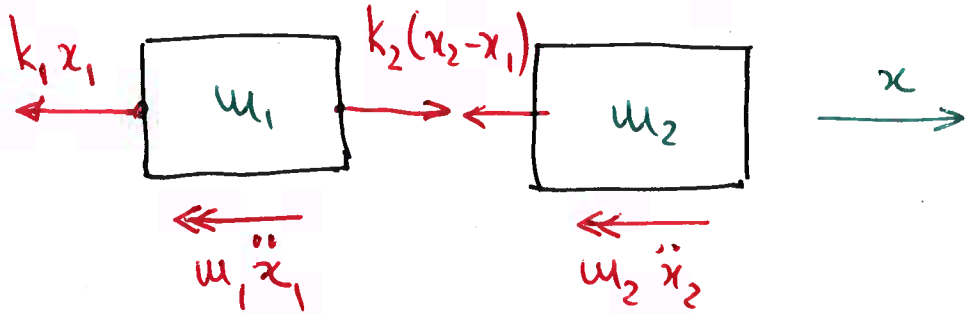
LM1

Lecture 2
24-Sep-2026



$$\dot{x} = \frac{dx}{dt}$$

$$\ddot{x} = \frac{d^2x}{dt^2}$$



$$\begin{cases} -k_1 x_1 + k_2(x_2-x_1) - m_1\ddot{x}_1 = 0 & \text{EoM } (x_1) \\ -k_2(x_2-x_1) - m_2\ddot{x}_2 = 0 & \text{EoM } (x_2) \end{cases}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$

kinetic energy $\frac{1}{2}mv^2$

potential energy (elastic) $\frac{1}{2}k\Delta x^2$

$$\begin{cases} T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \\ V = \left(\frac{1}{2}k_i \Delta x^2\right) = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2(x_2-x_1)^2 \end{cases}$$

1. Identify generalised coordinates (d.o.f.)
2. Write equation for total $T = \dots$
3. Write equation for total $V = \dots$
4. Use Lagrange equations

EoM x_1 $\quad q_i = x_1$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right)}_{\checkmark} - \underbrace{\frac{\partial T}{\partial x_1}}_{=0\ \checkmark} + \underbrace{\frac{\partial V}{\partial x_1}}_{} = 0$$

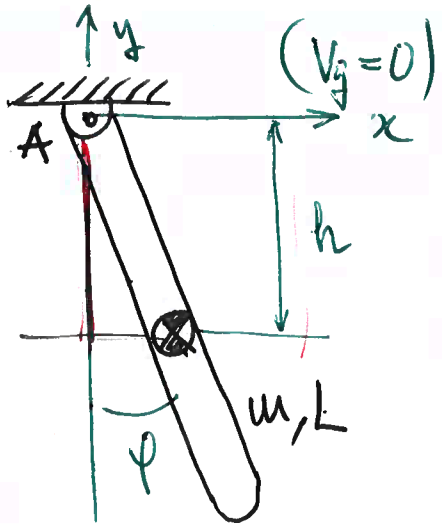
$$\frac{\partial T}{\partial \dot{x}_1} = \frac{2}{2} m_1 \dot{x}_1 = m_1 \dot{x}_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = \frac{d}{dt}\left(m_1 \dot{x}_1\right) = m_1 \ddot{x}_1$$

$$\frac{\partial V}{\partial x_1} = \frac{2}{2} k_1 x_1 + \frac{2}{2} k_2 (x_2 - x_1)(-1)$$

$$= k_1 x_1 - k_2 (x_2 - x_1)$$

$$\begin{cases} \text{EoM } x_1 \Rightarrow \boxed{m_1 \ddot{x}_1 + k_1 x_1 - k_2 (x_2 - x_1) = 0} \\ \text{EoM } x_2 \Rightarrow \boxed{m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0} \end{cases}$$



✓ (1.) find generalised coordinate
✓ (2.) find kinetic energy T
✓ (3.) find potential energy V
4. Lagrange equations

(2.) KE $T = \frac{1}{2} I_A \dot{\varphi}^2$ with $I_A = \frac{1}{3} mL^2$

$\Rightarrow T = \frac{1}{6} mL^2 \dot{\varphi}^2$

(3.) PE $V = V_e + V_g \left(= mgh\right) = -mg \frac{1}{2} L \cos\varphi$

V_e: 0 ; h: ?

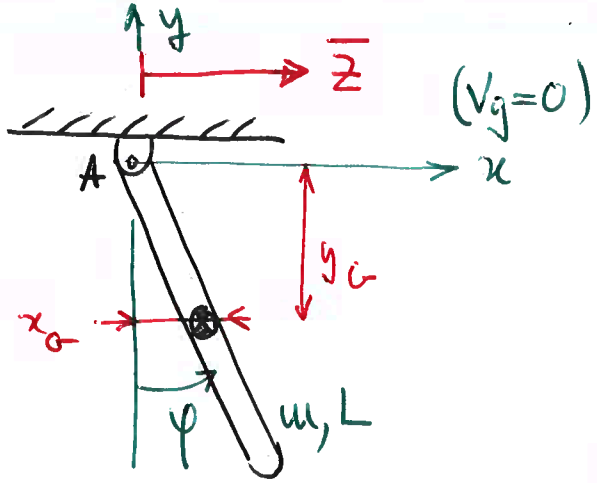
(4.) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi}^{0} + \frac{\partial V}{\partial \varphi} = 0$

$\frac{\partial T}{\partial \dot{\varphi}} = \frac{2}{6} mL^2 \dot{\varphi} \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) = \frac{1}{3} mL^2 \ddot{\varphi}$

$\frac{\partial V}{\partial \varphi} = +\frac{1}{2} mgL \sin\varphi$

EoM (φ): $\boxed{\frac{1}{3} mL^2 \ddot{\varphi} + \frac{1}{2} mgL \sin\varphi = 0}$ $\rightsquigarrow \varphi(t)$

inertia stiffness



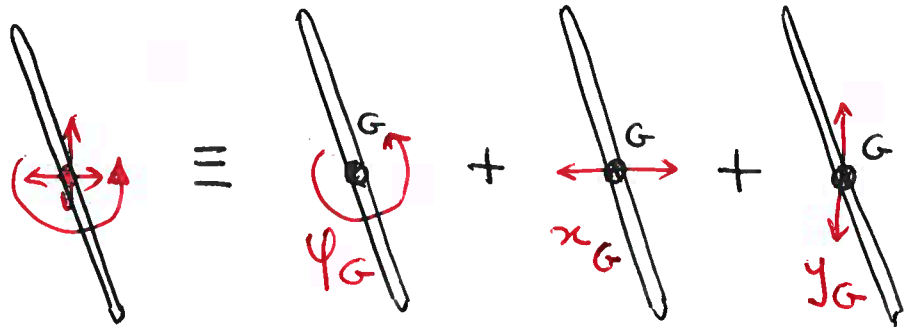
LM1

Lecture 4
1-Oct-2026

✓ 1. $\varphi(t)$
✓ 2. kinetic energy T
✓ 3. Potential energy V
4. Lagrang EoM

3. $V_g = -\frac{1}{2} mgl\cos\varphi$

2. $T = \dots$



$$T \equiv T_\varphi + T_{x_G} + T_{y_G}$$

$$T = \frac{1}{2} I_G \dot\varphi^2 + \frac{1}{2} m \dot x_G^2 + \frac{1}{2} m \dot y_G^2$$

$I_G = \frac{1}{12} ml^2$

$$\begin{cases} x_G = \frac{1}{2} l\sin\varphi + z \\ y_G = -\frac{1}{2} l\cos\varphi \end{cases} \Rightarrow \begin{cases} \dot x_G = \frac{1}{2} l\dot\varphi\cos\varphi + \dot z \\ \dot y_G = +\frac{1}{2} l\dot\varphi\sin\varphi \end{cases}$$

$$T = \frac{1}{24} ml^2\dot\varphi^2 + \frac{1}{2} m\left[\dot z^2 + \frac{1}{4} l^2\dot\varphi^2\cos^2\varphi + \dot z\, l\dot\varphi\cos\varphi + \frac{1}{4} l^2\dot\varphi^2\sin^2\varphi\right]$$

$\dot x_G^2$ $\qquad$ $\dot y_G^2$

$$\begin{cases} T = \frac{1}{24} m\ell^2 \dot{\varphi}^2 + \frac{1}{2} m \left[\dot{z}^2 + \frac{1}{4} \ell^2 \dot{\varphi}^2 + \dot{z} \ell \dot{\varphi} \cos\varphi \right] \\ V = -\frac{1}{2} mg\ell \cos\varphi \end{cases}$$

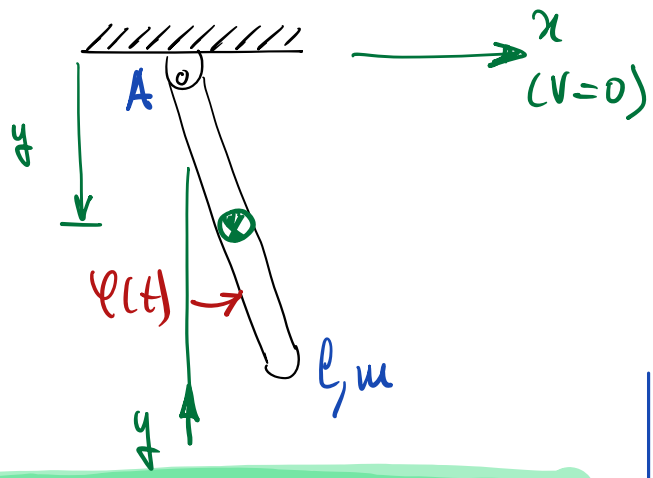
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi} + \frac{\partial V}{\partial \varphi} = 0$$

(...)

$$\boxed{\frac{1}{3} m\ell^2 \ddot{\varphi} + \frac{1}{2} m\ell \left(\ddot{z} \cos\varphi + g \sin\varphi \right) = 0} \rightsquigarrow \varphi(t) = \dots$$

$\cos\varphi \approx 1$ $\quad$ $\sin\varphi \approx \varphi$

$$\frac{1}{3} m\ell^2 \ddot{\varphi} + \frac{1}{2} m\ell \left(\ddot{z} + g\varphi \right) = 0$$



Solution based on pivot point A

$$T = \frac{1}{2} I_A \dot{\varphi}^2$$

$$V = mgy = -\frac{1}{2} l mg \cos\varphi$$

$$\frac{\partial T}{\partial \dot{\varphi}} = I_A \dot{\varphi}$$

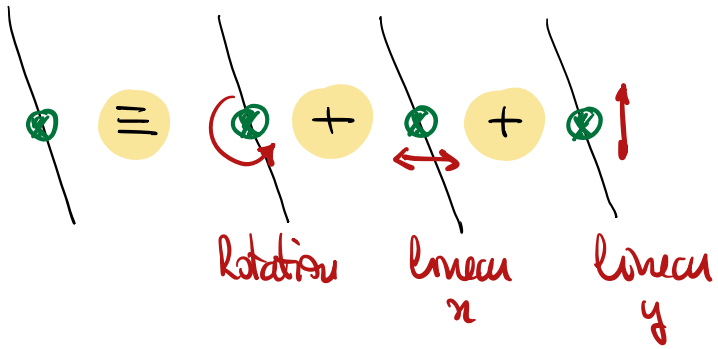
$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) = I_A \ddot{\varphi} = \frac{1}{3} ml^2 \ddot{\varphi}$$

$$\frac{\partial T}{\partial \varphi} = 0$$

$$\frac{\partial V}{\partial \varphi} = \frac{1}{2} mgl \sin\varphi$$

EoM: $\frac{1}{3} ml^2 \ddot{\varphi} + \frac{1}{2} mgl \sin\varphi$

Solution based on general planar motion



$$T = \frac{1}{2} I_{CG} \dot{\varphi}^2 + \frac{1}{2} m \left(\dot{x}_{CG}^2 + \dot{y}_{CG}^2\right)$$

$$\begin{cases} x_{CG} = \frac{1}{2} l \sin\varphi \\ y_{CG} = -\frac{1}{2} l \cos\varphi \end{cases} \quad \begin{cases} \dot{x}_{CG} = \frac{1}{2} l \dot{\varphi} \cos\varphi \\ \dot{y}_{CG} = \frac{1}{2} l \dot{\varphi} \sin\varphi \end{cases}$$

$$\begin{cases} \dot{x}_{CG}^2 = \frac{1}{4} l^2 \dot{\varphi}^2 \cos^2\varphi \\ \dot{y}_{CG}^2 = \frac{1}{4} l^2 \dot{\varphi}^2 \sin^2\varphi \end{cases}$$

$$\Rightarrow T = \frac{1}{2}\left(\frac{1}{12} ml^2\right) \dot{\varphi}^2 + \frac{1}{8} ml^2 \dot{\varphi}^2$$

$$V = -\frac{1}{2} mgl \cos\varphi$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) = \frac{1}{3} ml^2 \ddot{\varphi}$$

EoM: $\frac{1}{3} ml^2 \ddot{\varphi} + \frac{1}{2} mgl \sin\varphi$